Global hyperbolicity and Palais–Smale condition for action functionals in stationary spacetimes

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Dedicated to the memory of Professor Jerzy Konderak

Abstract

In order to apply variational methods to the action functional for geodesics of a stationary spacetime, some hypotheses, useful to obtain classical Palais–Smale condition, are commonly used: pseudo–coercivity, bounds on certain coefficients of the metric, etc. We prove that these technical assumptions admit a natural interpretation for the conformal structure (causality) of the manifold. As a consequence, any stationary spacetime with a complete timelike Killing vector field and a complete Cauchy hypersurface (thus, globally hyperbolic), is proved to be geodesically connected.

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1 Introduction

In the last years, an intensive research on the problem of geodesic connectedness in stationary spacetimes (i.e., the question whether any two points in a Lorentzian manifold admitting a timelike Killing vector field, can be joined by a geodesic) has been carried out. Even though there are geometric and physical reasons — no analog to Hopf–Rinow theorem exists for a Lorentzian manifold, stationary spacetimes include typical physical spacetimes, as Kerr's or Schwarzschild's — the main interest comes from the analytical viewpoint. In fact, given a Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$, geodesics connecting two fixed points $p, q \in \mathcal{M}$ are, among the C^1 curves connecting them, the critical points of the (energy) action functional

$$f(z) = \frac{1}{2}\int_0^1 \langle \dot z, \dot z\rangle_L \; ds, \tag{1.1}$$

which becomes strongly indefinite in the Lorentzian setting. Moreover, authors have mainly followed an extrinsic approach based on the assumption that the spacetime is *standard stationary*, i.e. $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ splits globally as $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$, with $(\mathcal{M}_0, \langle\cdot,\cdot\rangle)$ a finite dimensional connected Riemannian manifold, and metric $\langle\cdot,\cdot\rangle_L$ written as

$$\langle\cdot,\cdot\rangle_L = \langle\cdot,\cdot\rangle + 2\langle\delta(x),\cdot\rangle dt - \beta(x)dt^2 \tag{1.2}$$

for each $T_z\mathcal{M} \equiv T_x\mathcal{M}_0 \times \mathbb{R}$, $z = (x,t) \in \mathcal{M}$, where δ and β are a smooth vector field and a smooth strictly positive scalar field on $\mathcal{M}_0$, respectively. In this case, the lack of boundedness of f can be overcome by means of a suitable variational principle, stated in [4, 5, 13], which shows that looking for critical curves of action functional f connecting $p = (x_p, t_p)$ to $q = (x_q, t_q)$ becomes equivalent to the study of critical points for a new functional $\mathcal{J}$ on the Riemannian part, namely

$$\begin{aligned} \mathcal{J}(x) \;\; = \;\; & \frac{1}{2}\int_0^1 \langle \dot x, \dot x\rangle \; ds \; + \; \frac{1}{2}\int_0^1 \frac{\langle\delta(x), \dot x\rangle^2}{\beta(x)} \; ds \\ & - \frac{1}{2}\left(\int_0^1 \frac{\langle\delta(x), \dot x\rangle}{\beta(x)} \; ds - \Delta_t\right)^2 \left(\int_0^1 \frac{1}{\beta(x)} \; ds\right)^{-1} \end{aligned} \tag{1.3}$$

($\Delta_t = t_q - t_p$), which is defined on a suitable set of "spatial" curves joining x_p to x_q in $\mathcal{M}_0$ (for more details, see Proposition 4.1) and may also be bounded from below.

Since then, such a functional has been widely studied. Considering only the case $\langle\cdot,\cdot\rangle$ complete (without boundary), the known main results can be summarized as follows:

1. Benci, Fortunato and Giannoni [4, 5] studied the geodesic connectedness in a standard static spacetime ($\delta \equiv 0$) and introduced functional $\mathcal{J}$ for this case. Giannoni and Masiello [13] extended this study to the standard

stationary case. From these results (see also [17, Theorem 3.4.3]), the existence of critical points of $\mathcal{J}$ is ensured when β and $|\delta(x)|^2 = \langle \delta(x), \delta(x) \rangle$ have an upper bound and some $\epsilon \in \mathbb{R}$ exists so that

$$0 < \epsilon \leq \beta(x) \qquad \text{for all } x \in \mathcal{M}_0. \tag{1.4}$$

2. Pisani [21] used a different approach based on the direct study of action functional f. He obtained that under assumption (1.4) a sublinear growth for β and δ suffices for the existence of critical points, i.e., it is enough to assume that some $\alpha < 1$ exists so that

$$\beta(x), |\delta(x)| \leq \mu d^{\alpha}(x, \bar{x}) + k \quad \text{for all } x \in \mathcal{M}_0, \tag{1.5}$$

   where $d(\cdot, \cdot)$ is the canonical distance associated to $\langle \cdot, \cdot \rangle$, $\bar{x} \in \mathcal{M}_0$ is fixed, and $\mu \geq 0$, $k \in \mathbb{R}$ (see [21, Theorem 1.2] and also [10] for a multiplicity result).

3. Remarkably, Giannoni and Piccione [14] studied the existence of critical points for action f from a more intrinsic viewpoint. In principle, they assume only the existence of a complete timelike Killing vector field K. Then, for each $p, q \in \mathcal{M}$, they introduce a natural space of curves $C^1_K(p, q)$ associated to K, and consider the restriction of f to this space. Then, they define a notion of *pseudo–coercivity* for f, and show that, under this condition, p and q can be joined by a geodesic. However, there are two important limitations on this result: (a) pseudo–coercivity implies global hyperbolicity and, thus, the spacetime must be isometric to a standard stationary one (see next section), and (b) more unpleasantly, pseudo–coercivity is a very technical analytical condition. So, in order to give a more concrete result, they fix a Cauchy temporal function (which becomes equivalent to choose a splitting (1.2)), and reprove Pisani's result (at least when $\beta = -g(K, K)$ is bounded).

4. R. Bartolo and the authors [2] have applied very accurate estimates (some of them coming from [9]) to functional $\mathcal{J}$ in the standard static case, i.e., when the functional in (1.3) is simplified by $\delta \equiv 0$. As a consequence, the exponent $\alpha = 2$ in (1.5) is shown to be enough and optimal for the existence of critical points of $\mathcal{J}$. Recently, Bartolo, Candela and Flores [1] have extended this result to the stationary case, showing that it is sufficient to assume

$$0 \;<\; \beta(x) \;\leq\; \mu_1 d^2(x, \bar{x}) + k_1 \quad \text{for all } x \in \mathcal{M}_0, \tag{1.6}$$
$$|\delta(x)| \;\leq\; \mu_2 d(x, \bar{x}) + k_2 \quad \text{for all } x \in \mathcal{M}_0 \tag{1.7}$$

   (with $\bar{x} \in \mathcal{M}_0$ fixed, and $\mu_1, \mu_2 \geq 0$, $k_1, k_2 \in \mathbb{R}$). We must emphasize that these hypotheses correspond to the rough bounds in order to ensure the global hyperbolicity of the spacetime (see Appendix A.1).

The study of the standard stationary case hides an important fact: the same spacetime can split as (1.2) in very different ways (with very different β, δ) because just one such splitting is not intrinsic to the spacetime. As a simple and extreme example, Minkowski spacetime can be written as (1.2) either with an arbitrary growth of $|\delta|$ or with an incomplete $\langle \cdot, \cdot \rangle$ (see Appendix A.2). More deeply, the bounds for β, $|\delta|$ do not have a geometric meaning on $\mathcal{M}$, except as sufficient (but neither necessary nor intrinsic) conditions for global hyperbolicity.

Motivated by this type of objections, here we focus on Giannoni and Piccione's approach. The main limitation of their results is that pseudo–coercivity condition is analytical and very technical. In fact, it can be regarded as a tidy and neat version of Palais–Smale condition for the stationary ambient. But now the question is how to translate this technical condition in terms of the (Lorentzian) geometry of the manifold.

The aim of this paper is to answer this question by showing that, essentially, the geometrical meaning of pseudo–coercivity is *global hyperbolicity with a complete Cauchy hypersurface*. In fact, we prove the following result (which extends all the previous ones), discussing carefully all the hypotheses:

Theorem 1.1 *Let $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ be a stationary spacetime with a complete timelike Killing vector field K. If $\mathcal{M}$ is globally hyperbolic with a complete (smooth, spacelike) Cauchy hypersurface $\mathcal{S}$, then it is geodesically connected.*

Even more, well–known standard arguments in previous references (based on Ljusternik–Schnirelman category) allow one to prove also some multiplicity results when $\mathcal{M}$ is not contractible in itself. Concretely: (i) each two points $p, q \in \mathcal{M}$ can be joined by a sequence of spacelike geodesics $(z_n)_n$ with diverging $(f(z_n))_n$, (ii) given $p \in \mathcal{M}$ and an integral curve γ of K, the number of (future-directed) timelike geodesics which connect p and $\gamma(s)$, diverges when $s \to +\infty$.

This paper is organized as follows. In Section 2 we introduce the necessary Lorentzian tools, emphasizing the interplay between stationarity and global hyperbolicity. In Section 3 Giannoni–Piccione's intrinsic approach is revisited, and the relevant aspects for our problem are stressed. As the hypotheses of Theorem 1.1 will imply the existence of a splitting as in (1.2), in Section 4 we explain how the functional approach is simplified when one chooses one such splitting (but the results will be independent of the chosen one). The proof of Theorem 1.1 is carried out in Section 5. Previous discussion translates it into Theorem 5.1, and the crucial step for its proof is Proposition 5.2. Essentially, this proposition shows that, because of the existence of a complete Cauchy hypersurface, one must check Palais–Smale condition only for sequences of curves with bounded Riemannian norm. In the Appendix A we give exhaustive examples and discussions which show the accuracy of Theorem 1.1 and explain the meaning of the involved hypotheses. In general, we try to minimize the technicalities in the interplay between Causality Theory and Variational Methods (see Remark 3.3). Nevertheless, one of these technicalities, which regards the relation between continuous and H^1 causal curves is interesting in its own right, and is studied in Appendix B.

2 Tools in Lorentzian Geometry

In this section we briefly recall some basic notions in Lorentzian Geometry which will be used along the paper (for more details on Lorentzian manifolds, see [3, 15, 19, 20, 23]).

By a *Lorentzian manifold* $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ we mean a smooth[1] (connected) finite dimensional manifold equipped with a semi–Riemannian metric of index 1 on each tangent space $T_z\mathcal{M}$, $z \in \mathcal{M}$. A tangent vector $\zeta \in T_z\mathcal{M}$ is called *timelike* (respectively *lightlike*; *spacelike*; *causal*) if $\langle\zeta,\zeta\rangle_L < 0$ (respectively $\langle\zeta,\zeta\rangle_L = 0$ and $\zeta \neq 0$; $\langle\zeta,\zeta\rangle_L > 0$ or $\zeta = 0$; ζ is either timelike or lightlike). In what follows the Lorentzian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ will be also a *spacetime*, that is, $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is connected and time–orientable, with a prescribed time–orientation (a continuous choice of a causal cone at each $p \in \mathcal{M}$, which is called the *future* cone, in opposition to the non–chosen one or *past* cone).

A C^1 curve $\gamma : I \to \mathcal{M}$ (I real interval) is called timelike, lightlike, spacelike or causal when so it is $\dot\gamma(s)$ for all $s \in I$. For causal curves, this definition is extended to include piecewise C^1 curves: in this case, the two limit tangent vectors on the breaks must belong to the same causal cone. Accordingly, causal curves are called either *future* or *past* directed depending on the cone of $\dot\gamma(s)$.

A smooth curve $\gamma : I \to \mathcal{M}$ is a *geodesic* if its acceleration vanishes, i.e.,

$$\nabla^L_s \dot\gamma(s) = 0 \qquad \text{for all } s \in I, \tag{2.1}$$

where ∇^L_s denotes the covariant derivative along γ induced by the Levi–Civita connection of metric $\langle\cdot,\cdot\rangle_L$. In this case, the product

$$E_\gamma \equiv \langle\dot\gamma(s), \dot\gamma(s)\rangle_L \quad \text{for all } s \in I$$

is necessarily constant. The spacetime $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is *geodesically connected* if, given any two points $p, q \in \mathcal{M}$, there exists a geodesic $z^* : [0,1] \to \mathcal{M}$ such that $z^*(0) = p$ and $z^*(1) = q$. This property is equivalent to the existence of a critical point of the *action functional* defined in (1.1) in the set of all the C^1 curves $z : [0,1] \to \mathcal{M}$ such that $z(0) = p$ and $z(1) = q$ (and also in the extended domain of H^1 curves below).

A vector field K in $(\mathcal{M}, \langle\cdot,\cdot\rangle_L)$ is said *complete* if its integral curves are defined on the whole real line. On the other hand, K is said *Killing* if the Lie derivative of the metric tensor $\langle\cdot,\cdot\rangle_L$ with respect to K vanishes everywhere, or, equivalently, if the stages of all its local flows are isometries (i.e., $\langle\cdot,\cdot\rangle_L$ is invariant by its flow).

A well–known characterization of Killing vector fields is the following: K is a Killing vector field if and only if for each pair Y, W of vector fields, it is

$$\langle\nabla^L_Y K, W\rangle_L = -\,\langle\nabla^L_W K, Y\rangle_L.$$

[1] As a simplification, *smooth* will mean C^∞ as usual. But this hypothesis can be relaxed. In fact, one can assume that smooth means only C^4 for the spacetime and, consequently, C^3 for the elements which depend on first derivatives, as hypersurfaces. This allows one to apply Nash Embedding Theorem (in the spirit of most previous references on this topic) to some hypersurfaces, and no new problem will appear in our case, because the existence of smooth *Cauchy* hypersurfaces has been proved in [6] (see discussion below).

Thus, if $z : I \to \mathcal{M}$ is a C^1 curve and K is a Killing vector field it is

$$\langle \dot{z}, \nabla^L_s K(z)\rangle_L \equiv 0$$

on I. If z is only absolutely continuous, this holds almost everywhere in I. In particular, if z is a geodesic this property implies the existence of a constant $C_z \in \mathbb{R}$ such that

$$\langle \dot{z}, K(z)\rangle_L \equiv C_z \quad \text{for all } s \in I. \tag{2.2}$$

A spacetime is called *stationary* if it admits a timelike Killing vector field K. Locally, any stationary spacetime looks like a *standard stationary* one, i.e., the spacetime in (1.2). For these spacetimes, without loss of generality it can be assumed $K = \partial_t$ so to define the (future) time–orientation. If, in addition, K is also irrotational (i.e., its orthogonal distribution $K^\perp$ is involutive), the stationary spacetime is called *static*; in this case, locally it looks like a *standard static* one (i.e., its spacetime metric is the product one obtained in (1.2) with $\delta \equiv 0$). Notice that a static spacetime may be standard stationary but not standard static (see Remark 2.4 below).

Given p, $q \in \mathcal{M}$ the *causality relation* $p < q$ (respectively *chronological relation* $p \ll q$) means that there exists a future–directed causal (respectively timelike) curve from p to q. Denote by $p \leq q$ indistinctly either $p < q$ or $p = q$. Then, for each $p \in \mathcal{M}$ the *causal future* $J^+(p)$ and the *causal past* $J^-(p)$ are defined as

$$J^+(p) = \{q \in \mathcal{M} : p \leq q\} \qquad \text{and} \qquad J^-(p) = \{q \in \mathcal{M} : q \leq p\}.$$

Taking into account these relations, the space of piecewise C^1 causal curves can be extended in a way appropriate for convergence of curves (cf. [12, pp. 442] or also [3, pp. 54]):

Definition 2.1 A (non–necessarily smooth) *future-directed causal curve* $\gamma : I \to \mathcal{M}$ is a (continuous) curve which, for each convex[2] neighbourhood U, satisfies that, given $t, t' \in I, t < t'$ with $\gamma([t,t']) \subset U$, necessarily $\gamma(t) <_U \gamma(t')$, where $<_U$ is the causal relation in U, regarded as a spacetime (i.e., $\gamma(t)$ and $\gamma(t')$ can be joined by a future–directed C^1–causal curve contained entirely in U).

Remark 2.2 Causal curves, even if non–necessarily smooth, must be at least locally Lipschitzian and, thus, a.e. differentiable with finite integral of their length (see [20, Remark 2.26]). Notice that a continuous curve, a.e. differentiable, with timelike gradient (in the same time–orientation at each differentiable point) and finite integral of its length, is not necessarily a causal curve. A counterexample in Lorentz–Minkowski spacetime $\mathbb{L}^2$ can be constructed as follows. Consider a "devil's staircase" type function $t \in [0,1] \mapsto x(t) \in \mathbb{R}$ (the typical example is Cantor's function) which is continuous, with 0 derivative a.e., and connects $x(0) = 0$, $x(1) = 2$. Now, the curve in natural coordinates of $\mathbb{L}^2$,

[2] i.e., U is a (starshaped) normal neighbourhood of all its points (see [19, pp. 129]).

$\gamma(t) = (x(t), t)$, satisfies all the required properties, but it connects the non–causally related points $(0, 0), (2, 1)$.
Recall also that causal curves are absolutely continuous and, thus, they lie in the spaces of H^1 type defined below.

There are some equivalent definitions on what means to be globally hyperbolic for a spacetime:

(1) The spacetime is strongly causal (i.e., no "almost–closed" causal curves exist) and $J^+(p) \cap J^-(q)$ is compact for any $p, q \in \mathcal{M}$. Even more, it is worth pointing out that the assumption of being strongly causal can be weakened in only causal (absence of closed causal curves, see [18] for detailed explanations).

(2) The space of causal curves joining any two fixed points $p, q \in \mathcal{M}$ (defined from $[0, 1]$ to $\mathcal{M}$, but identified up to a strictly increasing monotonic reparametrization) is compact. The definition of the topology in such space of causal curves is somewhat subtle (see [16, 20]). Essentially, a priori we will exclude the existence of closed causal curves (otherwise, parametrizing one such a curve by giving more and more rounds, a sequence of non–equivalent causal curves would be obtained, and the compactness of the space of causal curves would be violated) and, then, the C^0 topology of curves is used.

(3) There exists a Cauchy hypersurface, that is, a subset which is crossed exactly once by any inextendible timelike curve.

A Cauchy hypersurface is necessarily a closed subset of $\mathcal{M}$ and an embedded topological hypersurface. A long–standing folk question has been if any globally hyperbolic spacetime must also admit a smooth Cauchy hypersurface which is spacelike (at all its points). Recently, this question has been answered affirmatively in [6] and, thus, we can take as a characterization of global hyperbolicity the existence of a *(smooth) spacelike Cauchy hypersurface* $\mathcal{S} \subset \mathcal{M}$. This characterization has the following remarkable consequence for stationary spacetimes:

Theorem 2.3 *A globally hyperbolic stationary spacetime is a standard stationary one, if some of its timelike Killing vector fields K is complete.*

Proof. Let $\mathcal{S}$ be a spacelike Cauchy hypersurface, and consider the map

$$\Psi : (x, t) \in \mathcal{S} \times \mathbb{R} \mapsto \Phi_t(x) \in \mathcal{M},$$

where Φ is the flow of the complete vector field K. As each point of $\mathcal{M}$ is crossed by one integral curve of K, which crosses $\mathcal{S}$ at exactly one point, Ψ is a diffeomorphism. As K is Killing, the pull–back metric $\Psi^*\langle\cdot, \cdot\rangle_L$ is independent of t and, thus, it makes $\mathcal{S} \times \mathbb{R}$ be a standard stationary spacetime. ∎

Remark 2.4 (1) If K is also irrotational, Theorem 2.3 does not yield the standard static splitting, as an integral manifold of $K^\perp$ may be non–Cauchy. A counterexample would be

$$\mathcal{M} = S^1 \times \mathbb{R}, \qquad \langle \cdot, \cdot \rangle_L = d\theta^2 + 2dtd\theta - dt^2,$$

where $(S^1, d\theta^2)$ is the standard unit circumference, and $K = \partial_t$.

(2) Function t on $\mathcal{M}$ obtained from Ψ^{-1} is a *Cauchy temporal function*, that is, the levels $t =$ constant are Cauchy hypersurfaces, and t is smooth with a past–directed timelike gradient (in particular, t is a time function, i.e. a continuous function which increases on any future–directed causal curve). As proved in [7], when such a temporal function exists the spacetime admits a global orthogonal splitting as in (1.2) with $\delta \equiv 0$ *but with* β *and* $\langle \cdot, \cdot \rangle$ *depending on* t. This splitting is obtained by flowing through the integral curves of ∇t (which, of course, are not equal to the integral curves of $K \equiv \partial_t$ in general) and, thus, it has a different nature from the splitting in Theorem 2.3.

(3) Under the hypotheses of Theorem 1.1, $\mathcal{S}$ can be chosen complete and, thus, so it is $\langle \cdot, \cdot \rangle$ in the splitting (1.2).

Finally, the following well–known property of globally hyperbolic spacetimes is stated for reference below (see, for example, [15, Proposition 6.6.6]):

Proposition 2.5 *If* $\mathcal{M}$ *admits a Cauchy hypersurface* $\mathcal{S}$ *then* $J^-(p) \cap \mathcal{S}$ *is compact, for any* $p \in \mathcal{M}$.

3 Abstract intrinsic functional framework

Throughout this section we will assume that $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ is a finite dimensional stationary spacetime with Killing vector field K. Next, geodesic connectedness of $(\mathcal{M}, \langle \cdot, \cdot \rangle_L)$ will be studied by using an intrinsic approach and, so, the framework introduced by Giannoni and Piccione in [14] is revisited.

In order to define notions as uniform convergence of curves or H^1 spaces, fix any auxiliary Riemannian metric $\langle \cdot, \cdot \rangle_R$ on $\mathcal{M}$. This metric can be chosen by leaving $\langle \cdot, \cdot \rangle_L$ unaltered on the orthogonal bundle of K, and reversing the sign on K, explicitly:

$$\langle \zeta, \zeta' \rangle_R = \langle \zeta, \zeta' \rangle_L \ - \ 2\, \frac{\langle \zeta, K(z) \rangle_L \ \langle \zeta', K(z) \rangle_L}{\langle K(z), K(z) \rangle_L} \tag{3.1}$$

for all ζ, $\zeta' \in T_z\mathcal{M}$, $z \in \mathcal{M}$ (this is the canonical choice in [14]). But recall that, on a standard stationary spacetime, i.e., when $\mathcal{M}$ is equipped with metric (1.2), metric (3.1) does not agree with $\langle \cdot, \cdot \rangle$ on $\mathcal{M}_0$, and may be incomplete on $\mathcal{M}$. Nevertheless, the results on this section are independent of the particular choice of $\langle \cdot, \cdot \rangle_R$.

As noticed in the previous section, the conservation law (2.2) is a natural constraint for geodesics in stationary spacetimes. Therefore, it results natural

to look for critical points of action functional f in (1.1) defined on the set of curves

$$C^1_K(p,q) = \{z \in C^1([0,1],\mathcal{M}) : \quad z(0)=p,\ z(1)=q, \text{ and } C_z \in \mathbb{R} \text{ exists such that } \langle \dot z, K(z)\rangle_L \equiv C_z\}.$$

As a first variational principle, we have (see [14, pp. 2][3]):

Theorem 3.1 *If $z \in C^1_K(p,q)$ is a critical point of f restricted to $C^1_K(p,q)$ then z is a geodesic connecting p and q.*

Even if functional f is defined in $C^1_K(p,q)$, it cannot be managed only in this space, as this space is "too small" for problems of convergence. So, the "natural" setting of this variational problem is a suitable submanifold of the space of H^1 curves from $[0,1]$ to $\mathcal{M}$, named $H^1([0,1],\mathcal{M})$.

Thus, we define the infinite dimensional manifold

$$\Omega^1(p,q) = \{z : [0,1] \to \mathcal{M} : \quad z \text{ is absolutely continuous and such that } z(0)=p,\ z(1)=q,\ \int_0^1 \langle \dot z, \dot z\rangle_R ds < +\infty\},$$

whose tangent space in each $z \in \Omega^1(p,q)$ can be identified with

$$T_z\Omega^1(p,q) = \quad \{\zeta : [0,1] \to T\mathcal{M} : \ \zeta(s) \in T_{z(s)}\mathcal{M} \text{ for all } s \in [0,1],\ \zeta \text{ is absolutely continuous and } \zeta(0) = 0 = \zeta(1),\ \|\zeta\|_* < +\infty\},$$

being its Hilbert norm

$$\|\zeta\|_*^2 = \int_0^1 \langle \nabla_s^R \zeta, \nabla_s^R \zeta\rangle_R ds,$$

where ∇_s^R denotes the covariant derivative along z relative to metric tensor $\langle\cdot,\cdot\rangle_R$ (nevertheless, we are not interested in its concrete value, which depends on the chosen $\langle\cdot,\cdot\rangle_R$, but only in the finiteness of the norm). Recall that functional f in (1.1) is well defined and finite on all $\Omega^1(p,q)$ (for example, notice that, for the choice (3.1) of $\langle\cdot,\cdot\rangle_R$, $\langle\zeta,\zeta\rangle_R \geq |\langle\zeta,\zeta\rangle_L|$). Even more, f is smooth with differential given by

$$f'(z)[\zeta] = \int_0^1 \langle \dot z, \nabla_s^L \zeta\rangle_L ds$$

for all $\zeta \in T_z\Omega^1(p,q)$, $z \in \Omega^1(p,q)$. Standard calculations allow one to prove that the critical points in $\Omega^1(p,q)$ are smooth curves which satisfy geodesic equation (2.1).

Analogously, the set $C^1_K(p,q)$ can be extended to a new subset of $\Omega^1(p,q)$ defined as

$$\Omega^1_K(p,q) = \{z \in \Omega^1(p,q) : \quad C_z \in \mathbb{R} \text{ exists such that } \langle \dot z, K(z)\rangle_L = C_z \text{ a.e. on } [0,1]\}. \tag{3.2}$$

[3] In [14, pp. 2] this results is actually stated for C^1 curves (not necessarily in $C^1_K(p,q)$). In order to pass to $C^1_K(p,q)$ a similar argument to that of [14, Proof of Theorem 3.3] is necessary.

In fact, standard arguments in Sobolev spaces imply that the closure of $C^1_K(p,q)$ is contained in $\Omega^1_K(p,q)$, furthermore $\Omega^1_K(p,q)$ is a C^2–submanifold of $\Omega^1(p,q)$ (see [14, Proposition 3.1]) whose tangent space in each point $z \in \Omega^1_K(p,q)$ is

$$T_z\Omega^1_K(p,q) = \{\zeta \in T_z\Omega^1(p,q) : \quad \begin{array}{l}\langle \nabla_s \zeta, K(z)\rangle_L + \langle \dot z, \nabla_\zeta K(z)\rangle_L \\ \text{is constant a.e. on } [0,1]\}\end{array}$$

(see [14, Corollary 3.2]). For simplicity, denote the restriction of action functional f on $\Omega^1_K(p,q)$ still with f. Thus, the following variational principle can be stated (for a complete proof, see [14, Theorem 3.3]).

Theorem 3.2 *A curve $z \in \Omega^1(p,q)$ is a geodesic in $\mathcal{M}$ if and only if $z \in \Omega^1_K(p,q)$ and z is a critical point of functional f in $\Omega^1_K(p,q)$.*

Remark 3.3 An essential step in the proof of Theorem 1.1 will be to construct causal curves from certain curves which connect two fixed points. If these curves belong to a general H^1–space, then we should extend the notion of C^1 causal curve to H^1 ones. When one makes this extension some subtleties appear (recall Remark 2.2), and the space of causal curves is reobtained (see Appendix B). Nevertheless, we will skip such technicalities by using sequences of curves in $C^1_K(p,q)$ and their limits in $\Omega^1_K(p,q) \subset \Omega^1(p,q)$.

Now, let us introduce the following definition which, essentially, translates classical "condition (C) of Palais–Smale" to our ambient.

Definition 3.4 *Fixed $c \in \mathbb{R}$ the set $\Omega^1_K(p,q)$ is c–precompact for f if every sequence $(z_n)_n \subset \Omega^1_K(p,q)$ with $f(z_n) \le c$ has a subsequence which converges weakly in $\Omega^1(p,q)$ (hence, uniformly in $\mathcal{M}$). Furthermore, the restriction of f to $\Omega^1_K(p,q)$ is pseudo–coercive if $\Omega^1_K(p,q)$ is c–precompact for all $c \ge \inf f(\Omega^1_K(p,q))$.*

The geodesic connectivity between each p and q will be a consequence of the following theorem (see [14, Theorem 1.2]).

Theorem 3.5 *If $C^1_K(p,q)$ is not empty and there exists $c > \inf f(C^1_K(p,q))$ such that $C^1_K(p,q)$ is c–precompact then there exists at least one geodesic joining p to q in $\mathcal{M}$.*

The assumption $C^1_K(p,q)$ non–empty must be imposed, because even if the stationary spacetime is globally hyperbolic it may not hold (see Appendix A.3 (a) for an explicit counterexample). Nevertheless, the possibility of $C^1_K(p,q) = \emptyset$ can be ruled out if K is complete (compare with [14, Lemma 5.7]).

Proposition 3.6 *Under the hypotheses of Theorem 2.3, for each p, $q \in \mathcal{M}$, it is $C^1_K(p,q) \neq \emptyset$.*

Proof. Let $p = (x_p, t_p)$, $q = (x_q, t_q) \in \mathcal{M}$ be fixed, and consider splitting (1.2) ensured by Theorem 2.3. As $\mathcal{M}$ and, thus, $\mathcal{S}$, is connected, a smooth curve $x : [0,1] \to \mathcal{S}$ exists joining x_p to x_q. Now, compute $t : [0,1] \to \mathbb{R}$ by imposing $t(0) = t_p$ and $\dot t = \frac{\langle \delta(x), \dot x\rangle - C}{\beta(x)}$ (i.e., $\langle (\dot x, \dot t), \partial_t\rangle_L \equiv C$), where constant C is chosen so to make $\int_0^1 \dot t ds = t_q - t_p$. ∎

4 The non–canonical global splitting

From now on, suppose that $\mathcal{M}$ has a complete timelike Killing vector field K and is globally hyperbolic with a complete spacelike Cauchy hypersurface $\mathcal{S}$. By Theorem 2.3, we can consider that the spacetime is the product $\mathcal{S}\times\mathbb{R}$, with the metric (1.2) for a certain vector field δ on $\mathcal{S}$ and the identifications

$$K(z) \equiv (0,1) \in T_x\mathcal{S}\times\mathbb{R} \quad \text{for all } z=(x,t)\in\mathcal{M},$$

$$\langle K(z), K(z)\rangle_L = -\beta(x) \quad \text{for all} \quad z=(x,t)\in\mathcal{M}.$$

Nevertheless, recall that neither K nor $\mathcal{S}$ are unique. Thus, this global splitting is not canonically associated to a spacetime under hypotheses of Theorem 1.1. Anyway, the results will be independent of the chosen $K,\mathcal{S}$.

For any absolutely continuous curve $z=(x,t):[0,1]\to\mathcal{M}$, it is

$$\langle \dot z(s), K(z(s))\rangle_L = \langle \delta(x(s)), \dot x(s)\rangle - \beta(x(s))\dot t(s) \quad \text{for a.e. } s\in[0,1]. \qquad (4.1)$$

Fixed $p=(x_p,t_p)$, $q=(x_q,t_q)\in\mathcal{M}$, it is

$$\Omega^1(p,q) \equiv \Omega^1(x_p,x_q;\mathcal{S})\times W(t_p,t_q),$$

where

$$\begin{aligned}\Omega^1(x_p,x_q;\mathcal{S}) = \quad & \{x:[0,1]\to\mathcal{S} : x \text{ is absolutely continuous and}\\ & \qquad x(0)=x_p,\ x(1)=x_q,\ \int_0^1 \langle \dot x,\dot x\rangle ds < +\infty\},\\ W(t_p,t_q) = \{t\in H^1([0,1],\mathbb{R}) &: t(0)=t_p,\ t(1)=t_q\} = H_0^1([0,1],\mathbb{R}) + T^*,\end{aligned}$$

with $H^1([0,1],\mathbb{R})$ classical Sobolev space and

$$H_0^1([0,1],\mathbb{R}) = \{t\in H^1([0,1],\mathbb{R}) : t(0)=t(1)=0\},$$

$$T^* : s\in[0,1] \longmapsto t_p + s\Delta_t \in\mathbb{R}, \quad \Delta_t = t_q - t_p.$$

Whence, $W(t_p,t_q)$ is a closed affine submanifold of $H^1([0,1],\mathbb{R})$ with tangent space $T_tW(t_p,t_q) = H_0^1([0,1],\mathbb{R})$ for all $t\in W(t_p,t_q)$. Moreover, it is

$$\begin{aligned}T_x\Omega^1(x_p,x_q;\mathcal{S}) = \quad & \{\xi:[0,1]\to T_x\mathcal{S} : \xi \text{ is absolutely continuous and}\\ & \qquad \xi(0)=\xi(1)=0,\ \int_0^1 \langle D_s\xi, D_s\xi\rangle ds < +\infty\}\end{aligned}$$

for all $x\in\Omega^1(x_p,x_q;\mathcal{S})$, where D_s denotes the covariant derivative along x induced by the Levi–Civita connection of metric $\langle\cdot,\cdot\rangle$.

Thus, taken any curve $z=(x,t)\in\Omega^1(p,q)$ it is

$$T_z\Omega^1(p,q) \equiv T_x\Omega^1(x_p,x_q;\mathcal{S})\times H_0^1([0,1],\mathbb{R})$$

and $\Omega^1(p,q)$ can be equipped with the Riemannian structure

$$\langle\zeta,\zeta\rangle_H = \langle(\xi,\tau),(\xi,\tau)\rangle_H = \int_0^1 \langle D_s\xi, D_s\xi\rangle\ ds \ + \ \int_0^1 \dot\tau^2 ds$$

for any $z = (x,t) \in \Omega^1(p,q)$ and $\zeta = (\xi,\tau) \in T_z\Omega^1(p,q)$.

By Nash Embedding Theorem, Riemannian hypersurface $\mathcal{S}$ can be assumed as a submanifold of a suitable Euclidean space $\mathbb{R}^N$ with $\langle\cdot,\cdot\rangle$ restriction to $\mathcal{S}$ of its Euclidean metric and $d(\cdot,\cdot)$ the corresponding distance. Furthermore, $\Omega^1(x_p,x_q;\mathcal{S})$ is a submanifold of classical Sobolev space $H^1([0,1],\mathbb{R}^N)$ and is complete because $\mathcal{S}$ is complete.

Clearly, (3.2) and (4.1) imply that $z = (x,t) \in \Omega^1_K(p,q)$ if and only if $x \in \Omega^1(x_p,x_q;\mathcal{S})$, $t \in W(t_p,t_q)$ and a constant $C_z \in \mathbb{R}$ exists such that

$$\langle\delta(x),\dot{x}\rangle - \beta(x)\dot{t} = C_z \qquad \text{a.e. on } [0,1]. \tag{4.2}$$

Hence, it is

$$\dot{t} = \frac{\langle\delta(x),\dot{x}\rangle - C_z}{\beta(x)} \qquad \text{a.e. on } [0,1], \tag{4.3}$$

which implies

$$C_z = \left(\int_0^1 \frac{\langle\delta(x),\dot{x}\rangle}{\beta(x)}\, ds \;-\; \Delta_t\right)\left(\int_0^1 \frac{ds}{\beta(x)}\right)^{-1}. \tag{4.4}$$

Moreover, by (4.3) it follows

$$\int_0^1 \beta(x)\dot{t}^2 ds = \int_0^1 \frac{\langle\delta(x),\dot{x}\rangle^2}{\beta(x)}\, ds - \; 2C_z\int_0^1 \frac{\langle\delta(x),\dot{x}\rangle}{\beta(x)}\, ds + C_z^2\int_0^1 \frac{ds}{\beta(x)},$$

and thus, the restricted action functional f becomes

$$\begin{aligned} f(z) &= \frac{1}{2}\int_0^1 \langle\dot{x},\dot{x}\rangle ds \;+\; C_z\Delta_t + \frac{1}{2}\int_0^1 \beta(x)\dot{t}^2\, ds \\ &= \frac{1}{2}\int_0^1 \langle\dot{x},\dot{x}\rangle ds \;+\; \frac{1}{2}\int_0^1 \frac{\langle\delta(x),\dot{x}\rangle^2}{\beta(x)}\, ds \;-\; \frac{C_z^2}{2}\int_0^1 \frac{ds}{\beta(x)}. \end{aligned} \tag{4.5}$$

In conclusion, now we can state the following variational principle introduced by Giannoni and Masiello in [13] for standard stationary spacetimes (see also [17, Theorem 3.3.2]):

Proposition 4.1 *A curve $z^* = (x^*,t^*) \in \Omega^1(p,q)$ is a critical point of action functional f in $\Omega^1(p,q)$ if and only if x^* is a critical point of functional*

$$\mathcal{J} : \Omega^1(x_p,x_q;\mathcal{S}) \to \mathbb{R}$$

defined in (1.3) and $t^* = \Psi(x^*)$ with $\Psi : \Omega^1(x_p,x_q;\mathcal{S}) \to W(t_p,t_q)$ such that

$$\begin{aligned} \Psi(x)(s) = t_0 + &\int_0^s \frac{\langle\delta(x(\sigma)),\dot{x}(\sigma)\rangle}{\beta(x(\sigma))}\, d\sigma \\ &- \left(\int_0^1 \frac{\langle\delta(x),\dot{x}\rangle}{\beta(x)}\, ds - \Delta_t\right)\int_0^s \frac{1}{\beta(x(\sigma))}\, d\sigma \left(\int_0^1 \frac{1}{\beta(x)}\, ds\right)^{-1}. \end{aligned} \tag{4.6}$$

Moreover, it is $f(z^*) = \mathcal{J}(x^*)$.

Remark 4.2 Taken any $z = (x,t) \in \Omega^1_K(p,q)$ formulae (1.3), (4.4) and (4.5) imply that $f(z) = \mathcal{J}(x)$ while (4.3), (4.4) and (4.6) imply $t = \Psi(x)$. In particular, let us point out that in the proof of Proposition 4.1 the introduced variational constraint is the kernel of the "partial derivative" operator

$$z \mapsto f'(z)[(\cdot, 0)]$$

which gives exactly the natural constraint (4.2).

5 Geodesic connectedness

The aim of the present section is to prove the following result.

Theorem 5.1 *Under the hypotheses of Theorem 1.1, the restriction of f to $C^1_K(p,q)$ is pseudo–coercive, for any p, $q \in \mathcal{M}$.*

Thus, Theorem 1.1 follows directly from the intrinsic result on geodesic connectedness for precompact $C^1_K(p,q)$ (see Theorem 3.5), the non–emptyness of $C^1_K(p,q)$ (see Proposition 3.6), and the coercivity ensured in Theorem 5.1.

The key step for the proof of Theorem 5.1 is the following proposition, which relates technically the global hyperbolicity of the spacetime with the notion of pseudo–coercivity. Let us remark that, in what follows, the hypotheses of Theorem 1.1 are always assumed, as well as the standard stationary splitting associated to the complete K and $\mathcal{S}$, and the notations introduced in Sections 3 and 4. In particular, for any curve x in $\mathcal{S}$, $\|\dot{x}\|^2 = \int_0^1 \langle \dot{x}, \dot{x}\rangle ds$, where the metric $\langle \cdot, \cdot \rangle$ is just the induced Riemannian metric on $\mathcal{S}$.

Proposition 5.2 *Let $(z_n)_n$, $z_n = (x_n, t_n)$, be a sequence of curves in $C^1_K(p,q)$ such that $f(z_n)(= \mathcal{J}(x_n))$ is upper bounded for all n. Then, $(\|\dot{x}_n\|)_n$ is bounded, too.*

Now, let us state some lemmas useful in the proof of Proposition 5.2 in which, for simplicity, we define $C^1(x_p, x_q; \mathcal{S}) = \Omega^1(x_p, x_q; \mathcal{S}) \cap C^1([0,1], \mathcal{M})$.

Firstly, let us point out that when the x_n's lie in a compact subset of $\mathcal{S}$, Proposition 5.2 is just a consequence of Cauchy–Schwarz inequality.

Lemma 5.3 *Let $(x_n)_n \subset C^1(x_p, x_q; \mathcal{S})$. If a compact subset $\mathcal{C}$ of $\mathcal{S}$ contains all the elements of the sequence $(x_n)_n$ and $\|\dot{x}_n\| \to +\infty$, then $\mathcal{J}(x_n) \to +\infty$.*

Proof. Consider definition (1.3). By expanding the squared term and using Cauchy–Schwarz inequality:

$$2\mathcal{J}(x_n) \geq \|\dot{x}_n\|^2 - \Delta_t \left(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds \right) \left(\int_0^1 \frac{ds}{\beta(x_n)} \right)^{-1} \tag{5.1}$$

for each $n \in \mathbb{N}$. Now, from the compactness of $\mathcal{C}$, there exist some strictly positive constants N_1, N_2, ν such that,

$$\nu \leq \beta(x_n(s)) \leq N_1,\ \langle \delta(x_n(s)), \delta(x_n(s)) \rangle \leq N_2, \text{ for all } s \in [0,1],\ n \in \mathbb{N}. \tag{5.2}$$

Thus, by applying again Cauchy–Schwarz, it is

$$2\mathcal{J}(x_n) \geq \|\dot{x}_n\|^2 - N\|\dot{x}_n\| - N'$$

for some $N, N' > 0$ independent of n, and the result follows. ∎

On the contrary, when no such compact $\mathcal{C}$ exists, Proposition 5.2 will be proved in two steps, Lemmas 5.5 and 5.6. But first notice that, given the spacelike parts $(x_n)_n$, a sequence of future–directed lightlike curves from $p = (x_p, t_p)$ to the integral curve of K through $q = (x_q, t_q)$ can be constructed. More precisely:

Lemma 5.4 *Fixed any $x \in C^1(x_p, x_q; \mathcal{S})$ (x non–constant if $x_p = x_q$) there exists a unique lightlike curve $\gamma^l = (x^l, t^l) : [0,1] \to \mathcal{M}$ joining (x_p, t_p) to $\{x_q\} \times \mathbb{R}$ in a time $T(x) = t^l(1) - t^l(0) > 0$ such that $x^l = x$. Moreover, $T(x)$ satisfies:*

$$T(x) = \int_0^1 \frac{\langle \delta(x), \dot{x} \rangle}{\beta(x)}\, ds \; + \int_0^1 \frac{\sqrt{\langle \delta(x), \dot{x} \rangle^2 + \langle \dot{x}, \dot{x} \rangle \beta(x)}}{\beta(x)}\, ds. \tag{5.3}$$

Proof. Fixing $x \in C^1(x_p, x_q; \mathcal{S})$, the t^l part of the curve $\gamma^l = (x, t^l) : [0,1] \to \mathcal{M}$ is characterized by the equalities $\langle \dot{\gamma}^l, \dot{\gamma}^l \rangle_L = 0$, $t^l(0) = t_p$ and the inequality $\dot{t}^l > 0$ a.e. on [0,1]. From (1.2) it follows

$$\dot{t}^l = \frac{\langle \delta(x), \dot{x} \rangle + \sqrt{\langle \delta(x), \dot{x} \rangle^2 + \langle \dot{x}, \dot{x} \rangle \beta(x)}}{\beta(x)} \quad \text{a.e. on [0,1]},$$

which implies directly (5.3). ∎

The global hyperbolicity of $\mathcal{M}$ becomes crucial for the first conclusion of the following result. The second one is just a simple consequence of the completeness of $\mathcal{S}$, but this property also turns out to be essential.

Lemma 5.5 *Let $(x_n)_n \subset C^1(x_p, x_q; \mathcal{S})$ and, for each $n \in \mathbb{N}$, denote $T_n = T(x_n)$. If no compact subset of $\mathcal{S}$ contains all the elements of sequence $(x_n)_n$, then, up to a subsequence, it is:*

(i) $T_n \to +\infty$, *and*

(ii) $\|\dot{x}_n\| \to +\infty$.

Proof. (i) Arguing by contradiction, let T^+ be an upper bound for all T_n, and put $p = (x_p, t_p)$, $q^+ = (x_q, T^+)$. The lightlike curves $\gamma_n^l = (x_n, t_n^l)$ obtained from Lemma 5.4, can be prolongued with the integral curve of K from (x_q, T_n) to (x_q, T^+); so, a piecewise smooth future–directed causal curve from p to q^+ is obtained and all the γ_n^l lie in $J^-(q^+)$. By Proposition 2.5, $J^-(q^+) \cap \mathcal{S}$ is compact, but this is a contradiction because this subset contains all the x_n's.
(ii) As $\mathcal{S}$ is complete, no bounded subset can contain all the x_n's. So, there is a sequence of points $(x_n(s_n))_n$ at arbitrary large distance from x_p, and the result follows. ∎

Lemma 5.6 *Fixed any sequence* $(x_n)_n \subset C^1(x_p, x_q; \mathcal{S})$ *such that*

$$\|\dot{x}_n\| \to +\infty \qquad \textit{and} \qquad T_n \to +\infty \tag{5.4}$$

then

$$\mathcal{J}(x_n) \to +\infty. \tag{5.5}$$

Proof. Let $(x_n)_n \subset C^1(x_p, x_q; \mathcal{S})$ be a sequence such that (5.4) holds and, for simplicity, let us assume $\Delta_t > 0$. Taking into account inequality (5.1), the desired limit (5.5) follows from (5.4), if a constant $k > 0$ exists such that

$$\left(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds\right)\left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1} \le k \quad \text{for all } n \in \mathbb{N}.$$

So, assume that, up to subsequences, it is

$$\left(\Delta_t - 2\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds\right)\left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1} \longrightarrow +\infty \quad \text{as } n \to +\infty. \tag{5.6}$$

On the other hand, by Cauchy–Schwarz inequality and definition (5.3), for each $n \in \mathbb{N}$ it is

$$T_n \le \tilde{T}_n \tag{5.7}$$

with

$$\begin{aligned}\tilde{T}_n \;=\;& \int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds \\ & + \sqrt{\left(\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle^2}{\beta(x_n)}\, ds + \|\dot{x}_n\|^2\right)\int_0^1 \frac{ds}{\beta(x_n)}}\end{aligned}$$

($\tilde{T}_n$ is the arrival time $\Delta_t \ge 0$ in the expression of $\mathcal{J}$ that we must choose in order to obtain $\mathcal{J}(x_n) = 0$; this arrival time is also useful in the context of Fermat principle as stated in [11]). Thus, it is

$$\begin{aligned}&\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle^2}{\beta(x_n)}\, ds + \|\dot{x}_n\|^2 \\ &\quad = \left(\tilde{T}_n - \int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds\right)^2 \left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1} \qquad \text{with } \tilde{T}_n \to +\infty.\end{aligned}$$

Hence, from (1.3) it follows

$$\begin{aligned}2\mathcal{J}(x_n) \;=\;& \left(\tilde{T}_n - \int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds\right)^2 \left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1} \\ & - \left(\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds \;-\; \Delta_t\right)^2 \left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1} \\ =\;& \left(\tilde{T}_n^2 - \Delta_t^2 - 2(\tilde{T}_n - \Delta_t)\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds\right)\left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1} \\ =\;& (\tilde{T}_n - \Delta_t)\left(\tilde{T}_n + \Delta_t - 2\int_0^1 \frac{\langle \delta(x_n), \dot{x}_n \rangle}{\beta(x_n)}\, ds\right)\left(\int_0^1 \frac{ds}{\beta(x_n)}\right)^{-1}.\end{aligned}$$

In conclusion, (5.5) follows from (5.4), (5.6) and (5.7). ■

Proof of Proposition 5.2.
The proof follows directly from Lemmas 5.3, 5.5, 5.6. ■

Proof of Theorem 5.1.
Let $p, q \in \mathcal{M}$ and $c \geq \inf f(C^1_K(p,q))$ be fixed, and consider a sequence $(z_n)_n$ in $C^1_K(p,q)$ such that

$$f(z_n) \leq c \quad \text{for all } n \in \mathbb{N}.$$

By the global splitting of $\mathcal{M}$, it is $p = (x_p, t_p)$ and $q = (x_q, t_q)$, while $z_n = (x_n, t_n)$ is such that $x_n \in C^1(x_p, x_q; \mathcal{S})$ and $t_n \in C^1([0,1], \mathbb{R}) \cap W(t_p, t_q)$ with

$$\langle \delta(x_n), \dot{x}_n \rangle - \beta(x_n)\dot{t}_n \equiv C_n \qquad \text{on } [0,1].$$

By Proposition 5.2,

$$(\|\dot{x}_n\|)_n \quad \text{has to be bounded,} \tag{5.8}$$

and all the x_n's lie in a bounded subset of $\mathcal{S}$. Thus, $(x_n)_n$ is bounded in $H^1([0,1], \mathbb{R}^N)$ and, up to subsequences, it converges to some $x \in H^1([0,1], \mathbb{R}^N)$ weakly in $H^1([0,1], \mathbb{R}^N)$ and uniformly in [0,1]. As $\mathcal{S}$ is complete, then $x \in \Omega^1(x_p, x_q; \mathcal{S})$, and all the x_n's lie in a compact subset. Thus, (5.2) holds, and, by (4.4) and Cauchy–Schwarz inequality, sequence $(C_n)_n$ has to be bounded. Hence, (4.3) and, again, Cauchy–Schwarz inequality imply that $(t_n)_n$ is bounded in $H^1([0,1], \mathbb{R})$ too, and thus, $t \in W(t_p, t_q)$ exists so that $t_n \to t$ uniformly in [0,1] (up to subsequences). ■

Appendix A: Discussion on the hypotheses and some counterexamples

§***1. Estimates for the global hyperbolicity of a standard stationary spacetime.*** As proven in [24], the spacetime (1.2) is globally hyperbolic if $\langle \cdot, \cdot \rangle$ is complete, β is at most quadratic and δ at most linear, that is (1.6) and (1.7) hold. This is a "rough estimate": it is easy to find counterexamples when the exponent of d in any of the inequalities is increased a bit, but these inequalities are only sufficient conditions. In fact, in the standard static case $\delta \equiv 0$, the spacetime is globally hyperbolic if and only if the conformal metric

$$\langle \cdot, \cdot \rangle^* = \frac{\langle \cdot, \cdot \rangle}{\beta(x)} \tag{5.9}$$

is complete (see [25] for more details). Notice that it is not relevant for $\langle \cdot, \cdot \rangle$ to be complete or not. In fact, classical Schwarzschild spacetime is globally hyperbolic with incomplete $\langle \cdot, \cdot \rangle$. Nevertheless, when a standard static spacetime is globally hyperbolic the slices at constant t are Cauchy hypersurfaces. Moreover, such a spacetime admits a *complete* spacelike Cauchy hypersurface $\mathcal{S}$ if and only if

$\langle\cdot,\cdot\rangle$ is complete (the non–trivial implication to the right can be proven because the projection $\mathcal{S}\to\mathcal{M}_0$ is a diffeomorphism which increases the distances).

§*2. **Arbitrariness of standard stationary splittings.*** As explained in the Introduction, essentially all the previous results in the literature on geodesic connectedness of stationary spacetimes rely in the behaviour of $\langle\cdot,\cdot\rangle,\beta,\delta$. Nevertheless, these elements are not canonical for the spacetime, in the sense that many such splittings are possible with a very different behavior for them. For example, $\mathbb{L}^2$ can be written as $\mathbb{R}^2$ endowed with the metric $dx^2+2\bar\delta(x)dxdt-dt^2$ for any function $\bar\delta$. This can be checked because the spacetime is a flat spaceform, i.e., its Gauss curvature is 0, it is simply connected and geodesically complete. The last property follows because, as a consequence of [22, Proposition 2.1], one has just to prove that the metric $\langle\cdot,\cdot\rangle_R$ in (3.1) is complete. But this is straightforward because, in the natural coordinates, the matrix of this metric has eigenvalues greater than a positive constant (see also [22, Example 2.4]). Thus, vector field $\delta=\bar\delta\partial_x$ may not satisfy the (at most) linear condition (1.7) above. Even more, it is also easy to construct incomplete Cauchy hypersurfaces in $\mathbb{L}^2$ and, then, by using the natural Killing vector field $K=\partial_t$, Theorem 2.3 yields a stationary splitting with incomplete $\langle\cdot,\cdot\rangle$.

§*3. **Accuracy of the hypotheses of Theorem 1.1.*** Let us check this with two counterexamples:

(a) *Stationary + Globally hyperbolic with complete $\mathcal{S}\not\Rightarrow$ geodesically connected.* Consider the spacetime obtained by removing in Lorentz–Minkowski $\mathbb{L}^{n+1}, n\geq 1$, the causal future of the points with $x_1=0=t$, in natural coordinates $(x_1,\ldots,x_n,t)$. Clearly, the spacetime admits the hyperplane $t\equiv -1$ as a complete Cauchy hypersurface, but it is not geodesically connected. Moreover, $C^1_K(p,q)=\emptyset$ for $p=(-1,0,\ldots,0), q=(1,0,\ldots,0)$.

(b) *Stationary with complete K + Globally hyperbolic $\not\Rightarrow$ geodesically connected.* Let $(\mathcal{S},\langle\cdot,\cdot\rangle)$ be a non–geodesically connected Riemannian manifold. Take β such that $\langle\cdot,\cdot\rangle^*$ in (5.9) is complete. Then, the standard static spacetime $(\mathcal{S}\times\mathbb{R},\langle\cdot,\cdot\rangle-\beta(x)dt^2)$ is globally hyperbolic and not geodesically connected, because the slice $t=0$ is totally geodesic (the family of static spacetimes given in [2, Section 7] also stresses the importance of global hyperbolicity).

§*4. **Accuracy of pseudo–coercivity from a technical viewpoint.*** As shown in Theorem 3.5, pseudo–coercivity of functional f (on $C^1_K(p,q)$ for all $p,\ q\in\mathcal{M}$) yields a technical natural condition for the geodesic connectedness of the spacetime. Let us discuss the relation between this condition and others involved in Theorem 1.1 as well as in [14]. Along this discussion, $(\mathcal{M},\langle\cdot,\cdot\rangle_L)$ is a stationary spacetime with a given timelike Killing vector field K, and we emphasize that, in the point (c), the Riemannian metric $\langle\cdot,\cdot\rangle_R$ on $\mathcal{M}$ will be chosen as the one associated to $\langle\cdot,\cdot\rangle_L$ and K by formula (3.1).

(a) *Functional f pseudo–coercive $\Rightarrow\mathcal{M}$ globally hyperbolic* (and, thus, the spacetime admits a splitting as standard stationary if K is complete). A

proof (valid in the case K complete) can be seen in [14, Proposition B.1]. An alternative argument based on the definitions of global hyperbolicity in Section 2 is the following. Arguing by contradiction, if the space of causal curves joining two points $p \leq q$ is not compact, then there exists a sequence $(z_n)_n$ of such future–directed causal curves with no converging subsequence. As each z_n can be approximated by piecewise smooth lightlike curves, we can assume that the z_n's are in fact lightlike curves. Thus, $f(z_n) = 0$ for any reparametrization of z_n. As $\langle \dot{z}_n, K(z)\rangle_L < 0$, we can choose this reparametrization (and smooth the possible finite number of breaks of z_n) in order to make z_n to belong to $C^1_K(p, q)$ with bounded $f(z_n)$. So, a converging subsequence of $(z_n)_n$ has to exist, which yields the contradiction.

Even more, if $\mathcal{M}$ is standard static then the pseudo–coercivity also implies $\langle \cdot, \cdot\rangle$ complete (notice that this does not hold in the stationary case, as explained at the end of §2). In fact, otherwise an incomplete geodesic $x : [0, 1) \to \mathcal{M}_0$ exists. The curve in $\mathcal{M}$, $z = (x, 0)$ will be a geodesic too, with $C_z = 0$. Now, consider the sequence of curves $z_n = (x_n, 0)$, where each x_n is a loop obtained by reparametrizing the restriction $x|_{[0,1-1/n]}$ in such a way that $p = x(0) = x_n(0) = x_n(1)$, $x_n(1/2) = x(1 - 1/n)$ for all $n > 1$. Again, $C_{z_n} = 0$, and $(z_n)_n$ violates the pseudo–coercivity of $C^1_K(p, p)$.

(b) *Functional f pseudo–coercive $\not\Rightarrow$ K complete.* A simple counterexample is any strip $\mathcal{M}_0 \times (a, b)$, $(a, b) \subsetneq \mathbb{R}$ of any standard static spacetime $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ with a complete Cauchy hypersurface (for example, $\mathcal{M} = \mathbb{L}^n$). In fact, f is pseudo–coercive for the full $\mathcal{M}$ (from the proof of Theorem 1.1), and the strip is still pseudo–coercive, as any curve $(x, t) \in C^1_K(p, q)$ has t either non–decreasing or non–increasing.

Nevertheless, recall that this is the unique example in the standard static case, and it does not yield new interesting examples in the stationary one. In fact, in the stationary case, when there exists a curve $(x, t) \in C^1_K(p, q)$ (for example, a geodesic) such that t admits either a strict maximum or minimum, then no strip $\mathcal{M}_0 \times (a, b)$ with $-\infty < a < b < +\infty$ is pseudo–coercive.

(c) *Functional f pseudo–coercive + K complete $\Rightarrow$ $\langle \cdot, \cdot\rangle_R$ in (3.1) complete* (in particular, $\langle \cdot, \cdot\rangle_R$ will be complete under the hypotheses of Theorem 1.1). From the final discussion in (a), the result is obvious in the standard static case, because then $\langle \cdot, \cdot\rangle$ becomes complete and $\langle \cdot, \cdot\rangle_R$ becomes the Riemannian warped product $\langle \cdot, \cdot\rangle_R = \langle \cdot, \cdot\rangle + \beta(x)dt^2$, which is complete, too (see [19, Lemma 7.40]). For the general case, choose any incomplete $\langle \cdot, \cdot\rangle_R$–geodesic $\gamma : I \to \mathcal{M}$ and consider the map $\psi : I \times \mathbb{R} \to \mathcal{M}$, $(s, t) \mapsto \Phi_t(\gamma(s))$, where Φ denotes the flow of K. Then, $I \times \mathbb{R}$, with the induced metric $\psi^*\langle \cdot, \cdot\rangle_L$, is static ($\partial_t$ is a timelike Killing vector field and, in dimension 2, any such vector field is irrotational). Moreover, $\psi^*\langle \cdot, \cdot\rangle_R$ coincides with the Riemannian metric on $I \times \mathbb{R}$ obtained from $\psi^*\langle \cdot, \cdot\rangle_L$ and ∂_t in (3.1). Recall also that f becomes pseudo–coercive for this spacetime;

thus, $I\times\mathbb{R}$ becomes globally hyperbolic and (as $I\times\mathbb{R}$ is 2–dimensional and simply connected) standard static. But, then, the standard static case is applicable, and the metric $\psi^*\langle\cdot,\cdot\rangle_R$ must be complete, a contradiction.

It is worth pointing out that the completeness of $\langle\cdot,\cdot\rangle_R$ implies the completeness of K (if an integral curve of K escapes any compact subset, it must have infinite length by the completeness of $\langle\cdot,\cdot\rangle_R$ and will be complete as it has constant $\langle\cdot,\cdot\rangle_R$–speed). Nevertheless, the completeness of $\langle\cdot,\cdot\rangle_R$ and global hyperbolicity are independent: any compact stationary spacetime is a counterexample for the implication to the right, and Schwarzschild spacetime is a counterexample for the converse.

Finally, we can wonder if both conditions together, the completeness of $\langle\cdot,\cdot\rangle_R$ and the global hyperbolicity of $\mathcal{M}$, would imply the existence of a complete Cauchy hypersurface (and, thus, geodesic connectedness, by Theorem 1.1). Nevertheless, this type of questions involves completely different techniques (see, for example, [8]) and goes beyond the scope of the present article.

Appendix B: H^1 causal curves are causal curves

The most general notion of causal curve is the one for *continuous* curves (see Definition 2.1). Necessarily, such a causal curve is H^1 (see Remark 2.2) with a.e. causal gradient, in the same time–orientation. Now, our purpose is to prove the converse, which becomes natural when properties of completeness of the set of causal curves are considered (see Remark 3.3). In the proof, the absolute continuity of H^1 curves will be essential (recall the Cantor–type counterexample in Remark 2.2). So, our final result can be stated as follows.

Theorem 5.7 *Let $(\mathcal{M},\langle\cdot,\cdot\rangle_L)$ be a spacetime, and $\gamma:[0,1]\to\mathcal{M}$, $I=[0,1]$, a continuous curve. Then, the following items are equivalent:*

- (i) *γ is future–directed causal (according to Definition 2.1);*
- (ii) *γ is H^1 and $\dot\gamma(s)$ is a future–directed causal vector for a.e. $s\in I$.*

Proof. According to Remark 2.2, we have only to prove that $(ii)\Rightarrow(i)$. Thus, taken γ so that (ii) holds, it is enough to show that, chosen any $s_0\in I$ and any convex neighborhood U of $z_0=\gamma(s_0)$, there exists $0<\delta<1$ such that if $0<s_1-s_0<\delta$ then $z_0<_U z_1=\gamma(s_1)$.
Firstly, notice that, in a neighbourhood $V\subseteq U$ of z_0, the spacetime can be written as $S\times(t_-,t_+)$ with

$$\langle\cdot,\cdot\rangle_L[x,t]=g_t[x]-dt^2,$$

where g_t is a Riemannian metric on S for any $t\in(t_-,t_+)$ and ∂_t is future–directed (for example, see [20, pp. 53]). Then, we can choose $\delta>0$ such that $\gamma([s_0,s_0+\delta])\subset V$.
Put $z_i=(x_i,t_i)$, $i=0,1$. Clearly, it is $z_0<_U z_1$ if there exists a C^1 curve $y:t\in[t_0,t_1]\mapsto y(t)\in S$ such that $y(t_i)=x_i$, $i=0,1$, and

$$g_t(\dot y(t),\dot y(t))\le 1 \qquad \text{for all } t\in[t_0,t_1]$$

(notice that $t \mapsto (y(t), t)$ would be future–directed causal).
To this aim, we need the following lemmas.

Lemma 5.8 *If there exists a sequence of C^1 curves $t \in [t_0, t_1] \mapsto y_n(t) \in S$ such that $y_n(t_i) = x_i$, $i = 0, 1$ and, for some sequence $\epsilon_n \searrow 0$,*

$$g_t(\dot{y}_n(t), \dot{y}_n(t)) \leq 1 + \epsilon_n \qquad \text{for all } t \in [t_0, t_1],$$

then $z_0 <_U z_1$.

Proof. The curves $t \mapsto (y_n(t), t_0 + \sqrt{1+\epsilon_n}(t - t_0))$ are future–directed and causal; thus, $z_0 <_U z_{1,n} = (x_1, t_0 + \sqrt{1+\epsilon_n}(t_1 - t_0))$. As U is convex, the relation $\leq_U$ is closed and the result follows passing to the limit as $n \to +\infty$. ∎

Lemma 5.9 *Writing the H^1 curve γ as $\gamma(s) = (x(s), t(s))$, if $s \in [s_0, s_0 + \delta]$, then function $s \mapsto t(s)$ is strictly increasing.*

Proof. As γ is absolutely continuous, then

$$t(s_3) = t(s_2) + \int_{s_2}^{s_3} \dot{t}(s) ds \qquad \text{for all } s_2,\ s_3 \in [s_0, s_0 + \delta].$$

But $\dot{\gamma}(s)$ is a.e. future–directed, so $\dot{t}(s) > 0$ a.e. in I; hence, $t(s_2) < t(s_3)$ if $s_2 < s_3$. ∎

By Lemma 5.9 it follows that the x–component of γ can be reparametrized by t, and this reparametrized curve still belongs to H^1. In fact, it is enough to prove that $t \mapsto x(t) \equiv x(s(t))$ satisfies Barrow's rule $x(t) = x(t_0) + \int_{t_0}^{t} (dx/dt)(\bar{t}) d\bar{t}$, and this is obvious because $dx/dt = (dx/ds)(ds/dt)$ a.e., the change of variable theorem for a.e. differentiable functions is applicable (see, for example, [26, Theorem 6.94]), and function $s \mapsto x(s)$ does satisfy Barrow's rule. Moreover, as $\dot{\gamma}$ is future–directed, it has to be

$$g_t\left(\frac{dx}{dt}, \frac{dx}{dt}\right) \leq 1 \qquad \text{a.e. in } [t_0, t_1]. \tag{5.10}$$

Now, take any sequence $(\tilde{y}_n)_n$ of C^1 curves in S with $\tilde{y}_n(t_i) = x_i$, $i = 0, 1$, which approach $t \mapsto x(t)$ in the H^1 norm. In particular, $(\tilde{y}'_n)_n$ also go to x' strongly in L^2 norm. In order to check that the sequence of C^1 curves $(y_n)_n$ formed by the reparametrizations of $\tilde{y}_n$ to constant speed, falls necessarily under the hypotheses of Lemma 5.8, firstly notice that the lengths $L_n = \text{length}(\tilde{y}_n) = \text{length}(y_n)$ satisfy $L_n \to L = \text{length}(x)$. Thus,

$$\left|\frac{dy_n}{dt}\right| = \frac{L_n}{t_1 - t_0} \to \frac{L}{t_1 - t_0} \leq 1$$

(the last inequality from (5.10)), and Lemma 5.8 applies.

∎

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